Analysis 2, Summer 2024

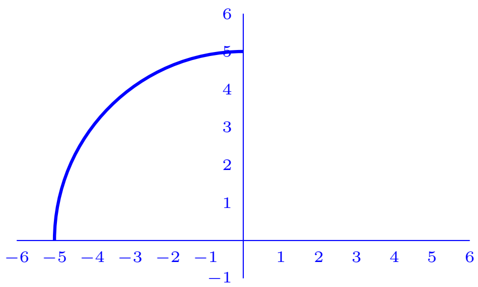
List 2

Path integrals, calculations with gradients

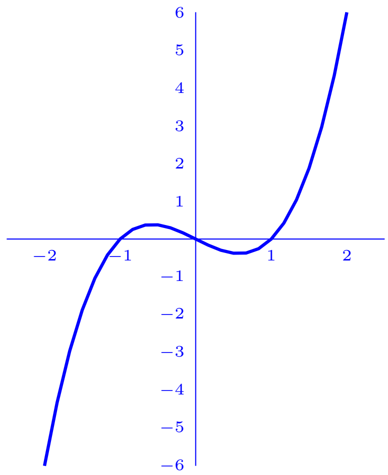
62. Draw the curve parameterized by

$$x = 5\cos(t), \qquad y = 5\sin(t)$$

with $\frac{\pi}{2} \leq t \leq \pi$. This is a quarter-circle.



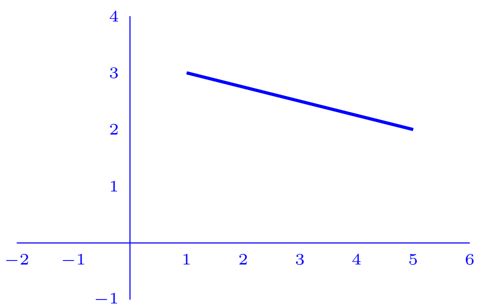
63. Draw the curve described by by $x = t,\ y = t^3 - t$ with $-2 \leq t \leq 2$. This is part of $y = x^3 - x$.



64. Draw the curve described by

$$\vec{r} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \end{bmatrix} t$$

with $0 \leq t \leq 1$. This is the line segment from $(1, 3)$ to $(5, 2)$.



A **parameterization** of a curve is a continuous vector function $\vec{r} : [a, b] \to \mathbb{R}^n$, where $[a, b]$ is some interval.

65. Give a parameterization of the line segment from that starts at $(0, 0)$ and ends at $(7, 2)$. One option is $\boxed{x = t,\ y = \frac{2}{7}t,\ 0 \le t \le 7}$. Another option is $\boxed{x = 7t,\ y = 2t,\ 0 \le t \le 1}$. There are many other correct parameterization.

66. Calculate $\int_C f \, \mathrm{d}s$ where $f(x, y) = x - y$ and C is the line segment that starts at $(2, 0)$ and ends at $(4, 5)$.

    Using $x = t, y = \frac{5}{2}t - 5, 2 \le t \le 4$, we get $\int_2^4 \frac{1}{4}\sqrt{29}(10 - 3t)\, \mathrm{d}t = \boxed{\frac{\sqrt{29}}{2}}$.

    Using $\vec{r} = (1 - t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \end{bmatrix}, 0 \le t \le 1$, we get $\int_0^1 \sqrt{29}(2 - 3t)\, \mathrm{d}t = \boxed{\frac{\sqrt{29}}{2}}$.

    Any other parameterization should also give this same answer.

67. Integrate $x - y$ along the line segment from $(2, 0)$ to $(4, 5)$. This is the same as the previous task.

68. Integrate xe^y along the half-circle $\{(x, y) : x^2 + y^2 = 1,\ x \ge 0\}$. $\boxed{e - \frac{1}{e}}$

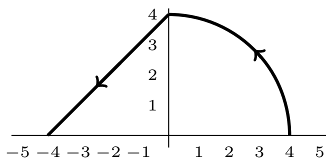
69. (a) Integrate $f(x, y) = \sin(\pi y)$ along the line segment from $(0, 1)$ to $(1, 0)$. $\boxed{\frac{2\sqrt{2}}{\pi}}$

    (b) Integrate $f(x, y) = \sin(\pi y)$ along the line segment from $(1, 0)$ to $(0, 1)$. $\boxed{\frac{2\sqrt{2}}{\pi}}$

    (c) Compare your answers to parts (a) and (b). They are the same. The "orientation" of a path does not affect the path integral of scalar function.

70. Calculate $\int_C (2yx^2 - 4x)\, \mathrm{d}s$ where C is the bottom half of the circle of radius 3 centered at the origin. $\boxed{-108}$

71. Integrate $f(x, y) = y$ along the path shown here:



    $\boxed{16 + 8\sqrt{2}}$

72. (a) Integrate $f(x, y, z) = xy + z$ along the "helix" curve $\vec{r} = [\cos(t), \sin(t), t]$ with $0 \le t \le 4\pi$. $\boxed{8\sqrt{2}\pi^2}$

    (b) Integrate $f(x, y, z) = xy + z$ along the line segment from $(1, 2, 3)$ to $(4, 5, 6)$. $\boxed{42\sqrt{3}}$

The **gradient** of $f(x, y)$ is the vector $\begin{bmatrix} f'_x \\ f'_y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$. We write ∇f for this vector.

73. Find the gradient of e^{x+y^2} at the point $(x,y)=(-5,2)$.

$f'_x(x,y) = e^{x+y^2}$ and $f'_y(x,y) = 2ye^{x+y^2}$, so $f'_x(-5,2) = e^{-5+4} = 1/e$ and $f'_y(-5,2) = 2(2)(1/e) = 4/e$. Therefore $\nabla f(-5,2) = \boxed{[1/e, 4/e]}$.

74. Calculate both $f(1,6)$ and $\nabla f(1,6)$ for $f(x,y) = x^4 + y\ln(x)$.

$f(1,6) = 1^4 + 6\ln(1) = 1 + 0 = \boxed{1}$.

$f'_x(x,y) = 4x^3 + \frac{y}{x}$ and $f'_y(x,y) = \ln(x)$, so $f'_x(1,6) = 4(1)^3 + \frac{6}{1} = 10$ and $f'_y(1,6) = \ln(1) = 0$. Therefore $\nabla f(1,6) = \boxed{[10, 0]}$.

75. Give the gradient of $\cos(x+y^2)$. (This will be a 2D vector whose entries are formulas with x and y.) $\boxed{\begin{bmatrix} -\sin(x+y^2) \\ -2y\sin(x+y^2) \end{bmatrix}}$

76. Compute the *length* of the gradient of $x^2\sin(y)$ at the point $(4, \frac{\pi}{3})$. (This is just a number.)

$\nabla f(4, \frac{\pi}{3}) = \begin{bmatrix} 4\sqrt{3} \\ 8 \end{bmatrix}$, so $|\nabla f(4, \frac{\pi}{3})| = \boxed{\sqrt{112} = 4\sqrt{7}}$.

77. Give ∇g for $g(x,y) = y^3\cos(xy) + \sqrt{x}$. $\boxed{\begin{bmatrix} \frac{1}{2\sqrt{x}} - y^4\sin(xy) \\ 3y^2\cos(xy) - xy^3\sin(xy) \end{bmatrix}}$

78. Calculate the gradient of $f(x,y,z) = xz + e^{y+z}$, which is defined as the 3D vector

$$\nabla f = \begin{bmatrix} f'_x \\ f'_y \\ f'_z \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}.$$

$\boxed{\begin{bmatrix} z \\ e^{y+z} \\ x + e^{y+z} \end{bmatrix}}$

79. For $f = \dfrac{x}{yz}$, calculate $\left|\nabla f(1,-1,2)\right|$. $\boxed{\dfrac{3}{4}}$

80. For $f(x,y) = \ln(x) + e^y$, calculate $(\frac{12}{13}\hat{\imath} + \frac{5}{13}\hat{\jmath}) \cdot \nabla f(4,0)$. $\begin{bmatrix} 12/13 \\ 5/13 \end{bmatrix} \cdot \begin{bmatrix} 1/4 \\ 1 \end{bmatrix} = \boxed{\dfrac{8}{13}}$.

81. For $f(x,y) = \dfrac{x}{y}$, give an example of a vector that is perpendicular to $\nabla f(12,2)$.

$\nabla f(12,2) = [\frac{1}{2}, -3]$, and a vector perpendicular to this is any non-zero scalar multiple of $\boxed{[3, \frac{1}{2}]}$, such as $\boxed{[6, 1]}$.

Starred tasks (☆) use ideas or methods that are not *required for this course. But they can be interesting to think about.*

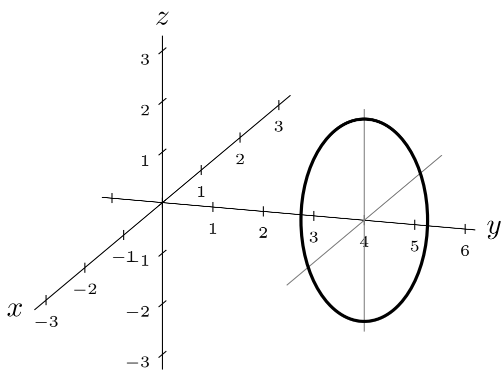
☆82. Find a function $f(x, y, z)$ for which $\nabla f = \begin{bmatrix} 2xz^3 - y\sin x \\ \cos x \\ 3x^2z^2 \end{bmatrix}$.

Any function $\boxed{f(x, y, z) = x^2z^3 + y\cos(x) + C}$ with C constant.

☆83. If $\vec{F} = \begin{bmatrix} x^3y \\ e^{yz} \\ y \end{bmatrix}$, calculate $\nabla \cdot \vec{F} = \boxed{3x^2y + ze^{yz}}$ and $\nabla \times \vec{F} = \boxed{(1 - ye^{yz})\hat{\imath} - x^3\hat{k}}$

using the idea that $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$.

☆84. Circles can be in 3D space! Integrate $f(x, y, z) = x^2y^2$ over the circle *in the vertical plane* $y = 4$ with center $(0, 4, 0)$ and radius 2. $\boxed{128\pi}$



See https://tutorial.math.lamar.edu/Solutions/CalcIII/LineIntegralsPtI/Prob5.aspx

☆85. The sets of points

$$A = \{(x, y) : x = \sin t,\ y = (\sin t)^2,\ 0 \le t \le \pi\}$$
$$B = \{(x, y) : x = \ln t,\ y = (\ln t)^2,\ 1 \le t \le e\}$$

are exactly the same (they are both $\{(x, y) : y = x^2, 0 \le x \le 1\}$). Why are

$$\int_0^\pi \sqrt{(x')^2 + (y')^2}\,\mathrm{d}t = \int_0^\pi \sqrt{(\cos t)^2 + (2\sin t\cos t)^2}\,\mathrm{d}t$$
$$\int_1^e \sqrt{(x')^2 + (y')^2}\,\mathrm{d}t = \int_1^e \sqrt{\left(\frac{1}{t}\right)^2 + \left(\frac{2\ln t}{t}\right)^2}\,\mathrm{d}t$$

not equal?

The description used for A traces out the path *twice* because the values of $x = \sin t$ go from 0 (when $t = 0$) to 1 (when $t = \frac{\pi}{2}$) and then back to 0 again (when $t = \pi$). If instead $0 \le t \le \frac{\pi}{2}$ is used then in fact

$$\int_0^{\pi/2} \sqrt{(\cos t)^2 + (2\sin t\cos t)^2}\,\mathrm{d}t = \int_1^e \sqrt{\left(\frac{1}{t}\right)^2 + \left(\frac{2\ln t}{t}\right)^2}\,\mathrm{d}t$$

are equal: they are both $\dfrac{2\sqrt{5} + \ln(2 + \sqrt{5})}{4}$.